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Tunneling from a Quantum Black Hole

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A quantum black hole has been presented by Kenmoku *et al.* (1998), and its surface gravity is divergent. We find that its tunneling probability is essentially different from Boltzmann distribution. It is interesting that two peaks appears in the spectrum when the black hole mass decreases close to Planck mass, which is different from black body radiation.

KEY WORDS: quantum black hole; tunneling; spectrum; surface gravity.
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Hawking radiation is an important landmark on the roads to quantum gravity. It implies that gravity has close relevance to statistical thermodynamics. It also opens a window for the incorporation of quantum theory and general relativity. Some problems, such as the information loss puzzle and the explanation of black hole entropy, are closely related to black hole radiation. Although 30 years past, the black hole thermodynamics is an important and lively realm in theoretical physics.

Strictly speaking, Hawking radiation is the semiclassical result because it does not involve the quantization of gravitational field. However, quantum fluctuations in spacetime may become crucial near the singularity and horizon, and then influences the Hawking radiation. Many efforts have been devoted to the quantization of black hole. It is interesting that the de Broglie-Bohm interpretation of quantum mechanics is applied to the quantum geometry of a Schwarzschild black hole (Kenmoku *et al.*, 1998). de Broglie-Bohm theory is not the standard interpretation of quantum mechanics. It, however, exhibits some advantages for the problems of time and observer in quantum gravity, because it needs no collapse of the wave function (Kenmoku *et al.*, 1998). In the context of this scenario, the geometry of a quantized black hole is given by

$$ds^2 = -\left(1 - \frac{2M}{v^{1/2}}\right) dT^2 + \left(1 - \frac{2M}{v^{1/2}}\right)^{-1} dr^2 + v d\Omega, \qquad (1)$$

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where

$$r = \frac{\pi}{2} \int z \left| H_\nu^{(2)}(z) \right|^2 d v^{1/2},$$
$$z = v_0 |v^{1/2} - 2M|. \tag{2}$$

This line element is obtained by quantizing the interior metric of black hole and exchanging the roles of time and spatial coordinates. For simplicity, we will set the constant $v_0 = 1$ in the following discussion. The horizon is located by

$$v^{1/2} = 2M. \tag{3}$$

z represents the quantum fluctuations of spacetime, $0 \leq z \leq \infty$. The classical geometry is obtained when $z \gg 1$. The smaller the value of z is (*i.e.*, the shorter the distance to horizon is), the more important quantum gravity effect becomes. In general, the black hole temperature is proportional to the surface gravity. This is because the period of the imaginary time, *i.e.*, the inverse temperature, is associated with the surface gravity by $\beta = 2\pi\kappa^{-1}$. However, direct calculation reveals that the surface gravity of the quantum black as shown in hole (1) is divergent, so are the other solutions of this kind (Wang and Liu, 2002; Gao and Shen, 2003). The divergent surface gravity means the vanishing period of imaginary time. This is an unusual case. We are curious to know what it implies.

Hawking radiation can be regarded as a tunneling process, similar to electron-positron pair creation in a constant electric field. However, most of the derivations of Hawking radiation are highly technical and cannot directly exhibit such a tunneling picture. There is an exception (Parikh and Wilczek, 2000), which provides a brief and direct derivation of Hawking effect, and reflects the tunneling nature of radiation. In this picture, the particle behind the horizon can tunnel out along a classically forbidden trajectory and the probability is given by

$$\Gamma \sim \exp(-2\mathbf{Im}S) \tag{4}$$

where S is the action for the trajectory. It is revealed that the tunneling probability is little different from the Boltzmann distribution. Hence, the black hole spectrum is not strictly thermal, and this difference may be crucial for resolving the puzzle of information loss. In this scenario, energy conservation plays a fundamental role and the self-gravitation of the emitted particle is considered. This tunneling method does not treat the black hole immersed in a thermal bath, and then a regular period of imaginary time is not necessarily required (*i.e.*, the surface gravity is not necessarily required to be regular). So, we are interested in applying this method to the quantum black hole presented by (1).

To remove the coordinate singularity, we introduce a new time coordinate t and do the following Painlevé type coordinate transformation

$$t = T + \int \frac{\sqrt{\frac{2M}{v^{1/2}}}}{1 - \frac{2M}{v^{1/2}}} dr. \tag{5}$$

Substituting it into the line element (1), we obtain

$$ds^2 = -\left(1 - \frac{2M}{v^{1/2}}\right) dt^2 + 2\sqrt{\frac{2M}{v^{1/2}}} dt\, dr + dr^2 + v d\Omega \tag{6}$$

Obviously, there is no singularity at the event horizon $v^{1/2} = 2M$. In order to get the spectrum of the particles from the quantum black hole, we consider the motion of the radial null geodesics on the geometry (6). The corresponding equation is

$$\dot{r} = \frac{dr}{dt} = \pm 1 - \sqrt{\frac{2M}{v^{1/2}}}, \tag{7}$$

where the plus and minus signs correspond to the outgoing and ingoing geodesics, respectively. When the self-gravitation effect of the emitted particle is considered by fixing the ADM mass of spacetime and letting the black hole mass vary, a shell with energy ω' travels on the outgoing geodesics of a spacetime with mass M replaced by $M' = M - \omega'$ (Parikh and Wilczek, 2000), namely,

$$d\tilde{s}^2 = -\left(1 - \frac{2M'}{v^{1/2}}\right) dt^2 + 2\sqrt{\frac{2M'}{v^{1/2}}} dtdr + dr^2 + v d\Omega. \tag{8}$$

The equation of motion of the null geodesics is modified as

$$\dot{r} = \frac{dr}{dt} = \pm 1 - \sqrt{\frac{2M'}{v^{1/2}}} \tag{9}$$

In the following discussion, we will take the plus sign because the outward light ray is considered. Since the wavelength of the outgoing particle is remarkably blue-shifted near the horizon, the WKB method is justified (Parikh and Wilczek, 2000). This approximation is also hidden in the original paper of Kenmoku *et al.* (1998), where the spacetime background is presented by (1). The light ray near the horizon is treated as a test particle and its motion is considered. The imaginary part of the action for a particle crossing the horizon from $r_{\rm in}$ to $r_{\rm out}$ is expressed as

$$\mathbf{Im} S = \mathbf{Im} \int_{\rm in}^{\rm out} p_r dr = \mathbf{Im} \int_{\rm in}^{\rm out} dr \int_0^{p_r} dp'_{r'} \tag{10}$$

where $r_{\rm in} = r|_{v^{1/2}=2M}$, $r_{\rm out} = r|_{v^{1/2}=2M'}$. Following from the Hamilton's equation

$$\dot{r} = \frac{\partial H}{\partial p'_r} = \frac{dM'}{dp'_r} \tag{11}$$

we obtain

$$\textbf{Im}S = \textbf{Im}\int_{\text{in}}^{out} dr \int \frac{dM'}{\dot{r}}$$
$$= \textbf{Im}\int_{r_{\text{in}}}^{r_{\text{out}}} dr \int_{M}^{M-\omega} \frac{dM'}{1-\sqrt{\frac{2M'}{v^{1/2}}}} \quad (12)$$

To ensure the modes of $\omega' > 0$ decay with time, we deform the contour into the upper half M' plane: $M' \to M' + i\epsilon$ (that is, into the lower half ω' plane: $\omega' \to \omega' - i\epsilon$). So, we have

$$\textbf{Im}S = \textbf{Im}\int_{r_{\text{in}}}^{r_{\text{out}}} dr(+i\pi)(-v^{1/2})$$
$$= -\pi \int_{r_{\text{in}}}^{r_{\text{out}}} v^{1/2} dr$$
$$= -\frac{\pi^2}{2}\int_{2M}^{2(M-\omega)} v^{1/2} z \left|H_\nu^{(2)}(z)\right|^2 dv^{1/2}$$
$$= \frac{\pi^2}{2}\int_{0}^{2\omega} z \left|H_\nu^{(2)}(z)\right|^2 (2M - z) dz, \quad (13)$$

where the last two equalities are due to the coordinate transformations presented by (2). For the classical case $z \gg 1$, the Hankel function

$$H_\nu^{(2)}(z) \to \exp(-iz)\sqrt{\frac{2}{\pi z}}, \quad (14)$$

and then

$$v^{1/2} \to r. \quad (15)$$

So, we obtain

$$\textbf{Im}S_0 = -\pi \int_{2M}^{2(M-\omega)} v^{1/2} dv^{1/2}$$
$$= 4\pi\omega\left(M - \frac{\omega}{2}\right), \quad (16)$$

which is the same as the argument of Parikh and Wilczek (2000), for a Schwarzschild black hole. We only consider $\nu = 0$ case, because $\nu \geq 1$ is not preferred by the Hermiticity of Hamitonian operator (Kenmoku *et al.*, 1998). Near the horizon $z \sim 0$, $H_0^{(2)}(z) \to 2(\ln z)/\pi$, so Eq. (13) becomes

$$\textbf{Im}S = \left(8M\omega^2 - \frac{16\omega^3}{3}\right)\left[(\ln 2\omega)^2 - \ln 2\omega + \frac{1}{2}\right] - \frac{16\omega^3}{3}\left(\frac{\ln 2\omega}{3} - \frac{5}{18}\right) \quad (17)$$

We still read out the imprint of Boltzmann distribution from (16). However, Eq. (17) is far different from Boltzmann distribution. We notice that the first power of ω is absent in Eq. (17). This might be the meaning of the divergent surface gravity of quantum black hole. Considering the tunneling probability $\Gamma \sim \exp(-2\mathbf{Im}S) < 1$, the distribution of energy-density reads (up to a factor of constant)

$$I(\omega) = \frac{\omega^3}{\exp(2\mathbf{Im}S) \pm 1} \tag{18}$$

where plus sign is for fermion and minus sign for boson. We first discuss the fermion. Essentially different from Planck spectrum of black body radiation, (18) will give a divergent density of energy

$$u \sim \int_0^\Lambda I(\omega)d\omega \to \infty, \tag{19}$$

if the upper limit Λ is arbitrarily large. This is because $\mathbf{Im}S$ becomes negative and cannot suppress the contribution of high frequency modes, when ω is sufficient large. Equation (16) is also confronted with the similar difficulty. So we must introduce a cutoff. A reasonable Λ should be of the order of Planck energy.[3] This cutoff cannot essentially modify the equation of state of black body radiation, because the contribution of the high frequency modes is rapidly suppressed by Boltzmann factor $\exp(-\beta\omega)$. For the black body radiation, the characteristic frequency is associated with the temperature by Wein's law. We also want to seek the characteristic frequency of the spectrum (18). Setting $I'(\omega) = 0$, we find that the characteristic frequency of $I(\omega)$ are determined by

$$3 - 32\omega^2(M - \omega)(\ln 2\omega)^2 = -3\exp(-2\mathbf{Im}S) \tag{20}$$

where $\mathbf{Im}S$ is given by Eq. (17). Since the formula shown in (17) is rather complicated, we cannot obtain the rigorous value of the characteristic frequency and the corresponding peak of $I(\omega)$. We can only learn the information of spectrum through graphics.

We see from Fig. 1 that the spectrum has only a characteristic frequency when black hole mass is much greater than Planck mass, and it is slowly blue-shifted when the mass decreases. Figure 2 is a 3D graphic, which exhibits that the corresponding peak increases with the decreasing mass.

Figure 3 exhibits how the characteristic frequency varies with the mass when the black hole evolves into the last stage. We notice that the second characteristic frequency (curve C2) appears when the black hole mass decreases close to Planck mass, and the first peak (C1) shifts to shorter wavelength. The curve $\overline{BC}$

[3] In fact, $\mathbf{Im}S > 0$ has been inquired in the derivation of spectrum (18), *i.e.*, the tunneling probability $\Gamma < 1$. This actually imposes a upper bound on the cutoff, $\Lambda \sim M$, which is looser for a large black hole. However, it may constrain the spectrum of bosons (see the later discussion about Fig. 8).

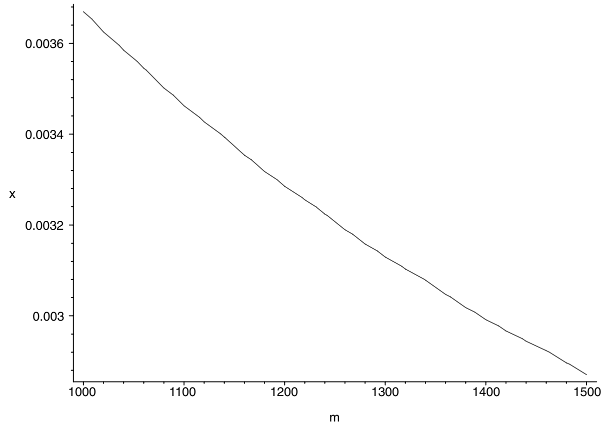

Fig. 1. The relation between the characteristic frequency and the mass of large black hole.

corresponds to the values of valley-bottom of $I(\omega)$. We notice that the first characteristic frequency vanishes (curve Cl ends at the point B) when the mass decreases to about three times of Planck mass. For more details, please look at Figs. 4–7.

Figures 4–7 exhibit how the second characteristic frequency and the corresponding peak evolve with the mass. The tendency is that both of them increase with the decreasing mass. When the mass goes down to about six times of Planck mass, the second peak appears (Fig. 4). Figure 5 exhibits that the second peak becomes comparable with the first one, and the spectrum is dominated by two peaks. When the mass further decreases to three times of Planck mass, the first

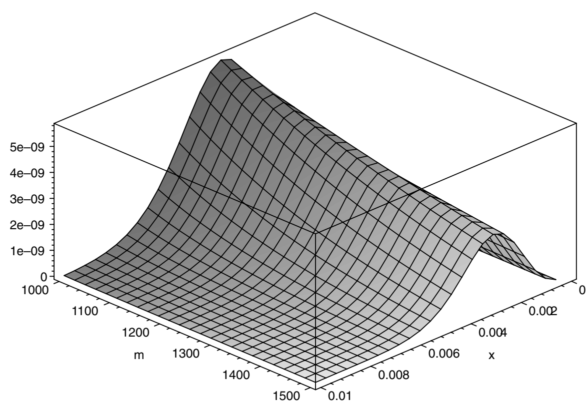

Fig. 2. 3D graphics for the spectrum of the large black hole. $x = 2\omega$, unmarked axes represents the distribution of energy density, $I(x)$.

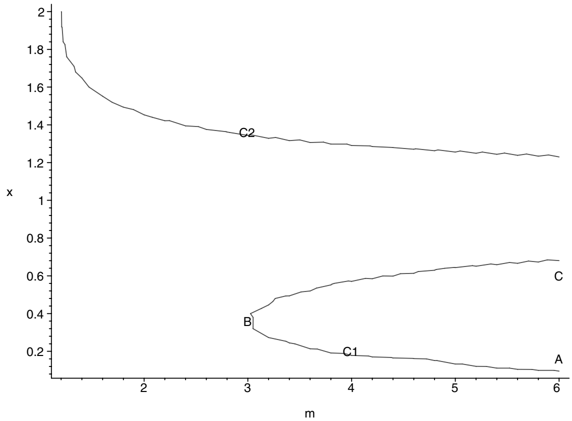

Fig. 3. The characteristic frequency varies with the mass of black hole at the last stage of evaporation.

peak vanishes and the second one takes the first place of the spectrum (Fig. 6). Figure 7 is a 3D graphic for the black hole at the last stage of evaporation, which exhibits that $I(\omega)$ increases with the decreasing mass rapidly. Although the first peak also increases, the second peak increases too fast and finally replaces the former.

The above graphics are in allusion to fermion. For the boson, we also obtain the similar scenario. The difference is that a sharp peak appears at the last stage

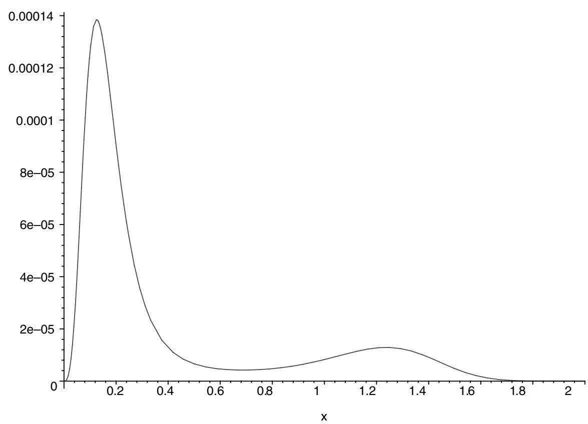

Fig. 4. The second peak of $I(\omega)$ begins to appear. Unmarked axes represents $I(x)$, the black hole mass is set to be 6 in this figure.

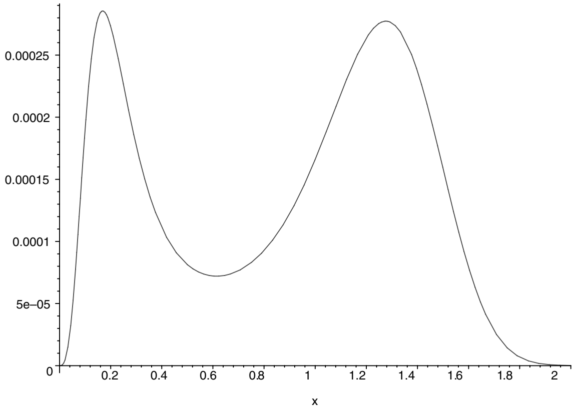

Fig. 5. The spectrum is dominated by two peaks. Unmarked axes represents $I(x)$, the black hole mass is set to be 4.5.

of an evaporating black hole (Fig. 8). This actually corresponds to $\mathbf{Im}S(\omega_s) = 0$. However, we find from (Fig. 8) that ω_s is larger than the black hole mass. Such a spectral line is probably nonexistent.

In summary, we discuss the tunneling scerario for the evaporation of a quantum black hole. The tunneling probability is essentially different from the semiclassical black holes. An interesting feature is that the spectrum possesses

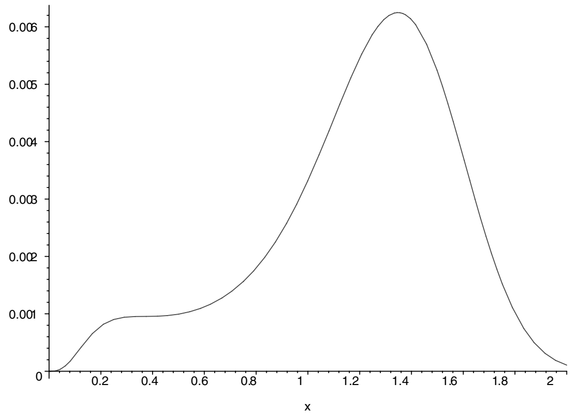

Fig. 6. The first peak vanishes and the second peak take the first place. Unmarked axes represents $I(x)$, the black hole mass is set to be 3.

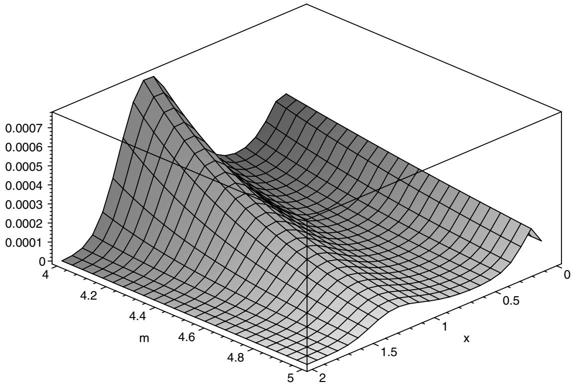

Fig. 7. 3D graphics for the spectrum which varies with the black hole mass at the last stage of evaporation. Unmarked axes represents $I(x)$.

two characteristic frequencies (corresponding to two peaks) when the black hole evolves into the last stage. The divergent surface gravity (*i.e.*, the absence of imaginary time period) means that the spectrum of the emitted particles seriously deviates from Boltzmann distribution. These results may imply that quantum geometry (1) is suitable to the last stage of an evaporating black hole.

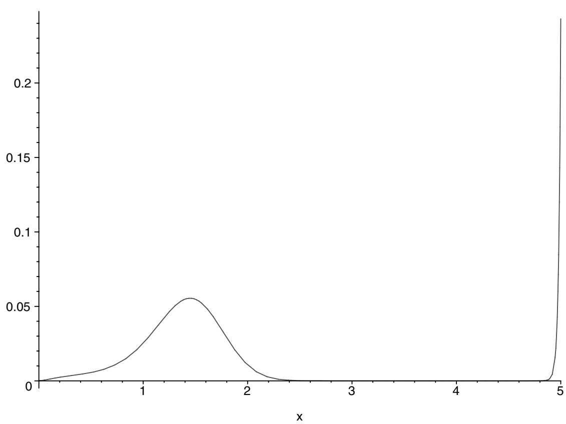

Fig. 8. The spectrum of the boson. A sharp peak appears at $x \approx 5(\omega_s \approx 2.5)$. the black hole mass is set to be 2. Unmarked axes represents $I(x)$.

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